

# CANONICAL BASIC SETS IN TYPE $B_n$

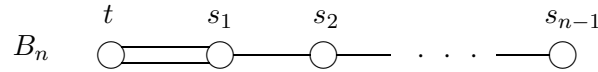
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*To Gordon James on his 60th birthday*

ABSTRACT. More than 10 years ago, Dipper, James and Murphy developed the theory of Specht modules for Hecke algebras of type  $B_n$ . More recently, using Lusztig's  $\mathbf{a}$ -function, Geck and Rouquier showed how to obtain parametrisations of the irreducible representations of Hecke algebras (of any finite type) in terms of so-called *canonical basic sets*. For certain values of the parameters in type  $B_n$ , combinatorial descriptions of these basic sets were found by Jacon, based on work of Ariki and Foda–Leclerc–Okado–Thibon–Welsh. Here, we consider the canonical basic sets for all the remaining choices of the parameters.

## 1. INTRODUCTION: SPECHT MODULES IN TYPE $B_n$

Let  $W_n$  be the finite Weyl group of type  $B_n$ , with generating set  $S_n = \{t, s_1, \dots, s_{n-1}\}$  and relations given by the following diagram:



We have  $W_n \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ , where  $\mathfrak{S}_n$  is the symmetric group on  $n$  letters. Let  $k$  be a field and  $Q, q \in k^\times$ . We denote by  $H_n := H_k(W_n, Q, q)$  the corresponding Iwahori–Hecke algebra. This is an associative algebra over  $k$ , with a basis  $\{T_w \mid w \in W_n\}$  such that the following relations hold for the multiplication:

$$\begin{aligned} T_t^2 &= Q T_1 + (Q - 1)T_t, \\ T_{s_i}^2 &= q T_1 + (q - 1)T_{s_i} \quad \text{for } 1 \leq i \leq n - 1, \\ T_w &= T_{r_1} \cdots T_{r_m}, \quad \text{for } w \in W_n, r_i \in S_n, m = \ell(w), \end{aligned}$$

where  $\ell: W_n \rightarrow \mathbb{N}$  is the usual length function. (We write  $\mathbb{N} = \{0, 1, 2, \dots\}$ .) In this paper, we are concerned with the problem of classifying the irreducible representations of  $H_n$ . For applications to the representation theory of finite groups of Lie type, see Lusztig's book [34] and the surveys [6], [16].

Hoefsmit [23] explicitly constructed the irreducible representations of  $H_n$  in the case where  $H_n$  is semisimple. In this case, we have a parametrization

$$\text{Irr}(H_n) = \{E^\lambda \mid \lambda \in \Pi_n^2\}.$$

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Here,  $\Pi_n^2$  denotes the set of all bipartitions of  $n$ , that is, pairs of partitions  $\lambda = (\lambda^{(1)} \mid \lambda^{(2)})$  such that  $|\lambda^{(1)}| + |\lambda^{(2)}| = n$ . In the general case where  $H_n$  is not necessarily semisimple, Dipper–James–Murphy [10] constructed a *Specht module*<sup>1</sup>  $S^\lambda$  for any bipartition  $\lambda \in \Pi_n^2$ . This is an  $H_n$ -module, reducible in general, such that  $\dim S^\lambda = \dim E^\lambda$ . Every Specht module comes equipped with an  $H_n$ -equivariant symmetric bilinear form. Taking the quotient by the radical, we obtain an  $H_n$ -module

$$D^\lambda := S^\lambda / \text{rad}(S^\lambda) \quad \text{for any } \lambda \in \Pi_n^2.$$

Let  $\Lambda_n^2 := \{\lambda \in \Pi_n^2 \mid D^\lambda \neq \{0\}\}$ . Then, by [10, §6], we have

$$\text{Irr}(H_n) = \{D^\lambda \mid \lambda \in \Lambda_n^2\}.$$

Through the work of Dipper–James–Murphy [10], Ariki–Mathas [4] and Ariki [2], [3], we have an explicit description of the set  $\Lambda_n^2$ . This depends on the following two parameters:

$$e := \min\{i \geq 2 \mid 1 + q + q^2 + \cdots + q^{i-1} = 0\}$$

$$f_n(Q, q) := \prod_{i=-(n-1)}^{n-1} (Q + q^i).$$

(We set  $e = \infty$  if  $1 + q + \cdots + q^{i-1} \neq 0$  for all  $i \geq 2$ .)

**Theorem 1.1.** *The set  $\Lambda_n^2$  is given as follows.*

- (A) (Dipper–James [8, 4.17 and 5.3] and Dipper–James–Murphy [10, 6.9]). *Assume that  $f_n(Q, q) \neq 0$ . Then*

$$\Lambda_n^2 = \{\lambda \in \Pi_n^2 \mid \lambda^{(1)} \text{ and } \lambda^{(2)} \text{ are } e\text{-regular}\}.$$

*(A partition is called  $e$ -regular if no part is repeated  $e$  times or more.)*

- (B) (Dipper–James–Murphy [10, 7.3]). *Assume that  $f_n(Q, q) = 0$  and  $q = 1$  (and, hence,  $Q = -1$ ). Then*

$$\Lambda_n^2 = \{\lambda \in \Pi_n^2 \mid \lambda^{(1)} \text{ is } e\text{-regular and } \lambda^{(2)} = \emptyset\}.$$

- (C) (Ariki–Mathas [4], Ariki [2]). *Assume that  $f_n(Q, q) = 0$  and  $q \neq 1$ . Thus,  $Q = -q^d$  where  $-(n-1) \leq d \leq n-1$ . Then*

$$\Lambda_n^2 = \{\lambda \in \Pi_n^2 \mid \lambda \text{ is a Kleshchev } e\text{-bipartition}\};$$

*this set only depends on  $e$  and  $d$ . (See Remark 4.7 where we recall the exact definition of Kleshchev bipartitions.)*

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<sup>1</sup>Actually, throughout this paper, we shall denote by  $S^\lambda$  the module that is labelled by  $(\lambda^{(2)*}, \lambda^{(1)*})$  in [10], where the star denotes the conjugate partition. For example, the index representation (given by  $T_t \mapsto Q$ ,  $T_{s_i} \mapsto q$ ) is labelled by the pair  $((n), \emptyset)$  and the sign representation (given by  $T_t \mapsto -1$ ,  $T_{s_i} \mapsto -1$ ) is labelled by  $(\emptyset, (1^n))$ . Thus, our labelling coincides with that of Hoefsmit [23].

Now, a fundamental feature of an Iwahori–Hecke algebra as above is that it can be derived from a “generic” algebra by a process of “specialisation”. For this purpose, let us assume that we can write

$$(\clubsuit) \quad Q = \xi^b \text{ and } q = \xi^a, \text{ where } \xi \in k^\times \text{ and } a, b \in \mathbb{N}.$$

Thus, the parameters  $Q, q$  of  $H_n$  are assumed to be integral powers of one fixed non-zero element  $\xi$  of  $k$ . (Note that  $\xi, a, b$  are not uniquely determined by  $Q, q$ .) This situation naturally occurs, for example, in applications to the representation theory of reductive groups over a finite field  $\mathbb{F}_q$ , where  $\xi = q1_k$ . The integers  $a, b$  uniquely define a weight function  $L: W_n \rightarrow \mathbb{Z}$  in the sense of Lusztig [36], that is, we have

$$\begin{aligned} L(t) &= b, & L(s_1) &= \cdots = L(s_{n-1}) = a, \\ L(ww') &= L(w) + L(w') & \text{whenever } \ell(ww') &= \ell(w) + \ell(w'). \end{aligned}$$

Let  $R = \mathbb{Z}[u, u^{-1}]$  be the ring of Laurent polynomials in an indeterminate  $u$ . Let  $\mathbf{H}_n = \mathbf{H}_R(W_n, L)$  be the corresponding generic Iwahori–Hecke algebra. This is an associative algebra over  $R$ , with a free  $R$ -basis  $\{\mathbf{T}_w \mid w \in W_n\}$  such that the following relations hold for the multiplication:

$$\begin{aligned} \mathbf{T}_t^2 &= u^b \mathbf{T}_1 + (u^b - 1) \mathbf{T}_t, \\ \mathbf{T}_{s_i}^2 &= u^a \mathbf{T}_1 + (u^a - 1) \mathbf{T}_{s_i} & \text{for } 1 \leq i \leq n-1, \\ \mathbf{T}_w &= \mathbf{T}_{r_1} \cdots \mathbf{T}_{r_m}, & \text{for } w \in W_n, r_i \in S_n, m = l(w). \end{aligned}$$

Then there is a ring homomorphism  $\theta: R \rightarrow k$  such that  $\theta(u) = \xi$ , and we obtain  $H_n$  by extension of scalars from  $R$  to  $k$  via  $\theta$ :

$$H_n = k \otimes_R \mathbf{H}_R(W_n, L).$$

Now, it would be desirable to obtain a parametrization of the irreducible representations of  $H_n$  which also takes into account the weight function  $L$ . (Note that  $\Lambda_n^2$  is “insensitive” to  $L$ : it does not depend on the choice of  $a, b, \xi$  such that  $(\clubsuit)$  holds.) It would also be desirable to obtain a parametrization which fits into a general framework valid for Iwahori–Hecke algebras of any finite type. (Note that it is not clear how to define Specht modules for algebras of exceptional type, for example.) Such a general framework for obtaining  $L$ -adapted parametrizations was developed by Geck [12], [13], [16] and Geck–Rouquier [21]. It relies on deep (and conjectural for general choices of  $L$ ) properties of the *Kazhdan–Lusztig basis* and Lusztig’s *a-function*. In this framework, the parametrization is in terms of so-called “canonical basic sets”. We recall the basic ingredients in Section 2.

Jacon [24], [25], [26] explicitly described these canonical basic sets in type  $B_n$  for certain choices of  $a$  and  $b$ , most notably the case where  $a = b$  (the “equal parameter case”) and the case where  $b = 0$  (which gives a classification of the irreducible representations of an algebra of type  $D_n$ ). Note that, in these cases, the canonical basic sets are different from the set  $\Lambda_n^2$ . It is known, see Theorem 2.8, that  $\Lambda_n^2$  can be interpreted as a canonical basic set with respect to weight functions  $L$  such that  $b > (n-1)a > 0$ . The

aim of this article to determine the canonical basic sets for all the remaining choices of  $a, b$ . This goal will be achieved in Theorems 3.1, 3.4 and 5.4; note, however, that for ground fields of positive characteristic, our solution in case (C) of Theorem 1.1 relies on the validity of Lusztig's conjectures [36] on Hecke algebras with unequal parameters.

An application of the results in this paper to the modular representation theory of finite groups of Lie type can be found in [17]: The explicit description of canonical basic sets for  $H_n$  yields a natural parametrization of the modular principal series representations for finite classical groups.

## 2. LUSZTIG'S $\mathbf{a}$ -FUNCTION AND THE DECOMPOSITION MATRIX

We keep the setting of the previous section, where  $\mathbf{H}_n = \mathbf{H}_R(W_n, L)$  is the generic Iwahori–Hecke algebra corresponding to the Weyl group  $W_n$  and the weight function  $L$  such that  $L(t) = b \geq 0$  and  $L(s_i) = a \geq 0$  for  $1 \leq i \leq n-1$ . The aim of this section is to recall the basic ingredients in the definition of a *canonical basic set* for the algebra  $H_n = k \otimes_R \mathbf{H}_n$  where  $\theta: R \rightarrow k$  is a ring homomorphism into a field  $k$  and  $\xi = \theta(u)$ .

At the end of this section, in Theorem 2.8, we recognise the set  $\Lambda_n^2$  (arising from the theory of Specht modules) as a canonical basic set for a certain class of weight functions.

Let  $K = \mathbb{Q}(u)$  be the field of fractions of  $R$ . By extension of scalars, we obtain a  $K$ -algebra  $\mathbf{H}_{K,n} = K \otimes_R \mathbf{H}_n$ , which is known to be split semisimple; see Dipper–James [8]. Furthermore, in this case, we have

$$\mathrm{Irr}(\mathbf{H}_{K,n}) = \{S^\lambda \mid \lambda \in \Pi_n^2\}$$

where  $S^\lambda$  are the Specht modules defined by Dipper–James–Murphy [10]. (Recall our convention about the labelling of these modules.)

The definition of Lusztig's  $\mathbf{a}$ -function relies on the fact that  $\mathbf{H}_n$  is a symmetric algebra. Indeed, we have a trace form  $\tau: \mathbf{H}_n \rightarrow R$  defined by

$$\tau(\mathbf{T}_1) = 1 \quad \text{and} \quad \tau(\mathbf{T}_w) = 0 \quad \text{for } w \neq 1.$$

The associated bilinear form  $\mathbf{H}_n \times \mathbf{H}_n \rightarrow R$ ,  $(h, h') \mapsto \tau(hh')$ , is symmetric and non-degenerate. Thus,  $\mathbf{H}_n$  has the structure of a *symmetric algebra*. Now extend  $\tau$  to trace form  $\tau_K: \mathbf{H}_{K,n} \rightarrow K$ . Since  $\mathbf{H}_{K,n}$  is split semisimple, we have

$$\tau_K(T_w) = \sum_{\lambda \in \Pi_n^2} \frac{1}{\mathbf{c}_\lambda} \mathrm{trace}(\mathbf{T}_w, S^\lambda) \quad \text{for all } w \in W_n,$$

where  $0 \neq \mathbf{c}_\lambda \in R$ ; see [20, Chapter 7]. We have

$$\mathbf{c}_\lambda = f_\lambda u^{-\mathbf{a}_\lambda} + \text{combination of higher powers of } u,$$

where both  $f_\lambda$  and  $\mathbf{a}_\lambda$  are integers,  $f_\lambda > 0$ ,  $\mathbf{a}_\lambda \geq 0$ .

Hoefsmit [23] obtained explicit combinatorial formulas for  $\mathbf{c}_\lambda$ . Then Lusztig deduced purely combinatorial expressions for  $f_\lambda$  and  $\mathbf{a}_\lambda$ ; see [36,

Chap. 22]. Let us first assume that  $a > 0$ . Then we have

$$\begin{aligned} f_{\lambda} &= 1 && \text{if } a > 0 \text{ and } b/a \notin \{0, 1, \dots, n-1\}, \\ f_{\lambda} &\in \{1, 2, 4, 8, 16, \dots\} && \text{if } a > 0 \text{ and } b/a \in \{0, 1, \dots, n-1\}; \end{aligned}$$

see [36, 22.14]. To describe  $\mathbf{a}_{\lambda}$ , we need some more notation. Let us write

$$b = ar + b' \quad \text{where } r, b' \in \mathbb{N} \text{ and } b' < a.$$

Let  $\lambda = (\lambda^{(1)} \mid \lambda^{(2)}) \in \Pi_n^2$  and write

$$\lambda^{(1)} = (\lambda_1^{(1)} \geq \lambda_2^{(1)} \geq \lambda_3^{(1)} \geq \dots), \quad \lambda^{(2)} = (\lambda_1^{(2)} \geq \lambda_2^{(2)} \geq \lambda_3^{(2)} \geq \dots),$$

where  $\lambda_N^{(1)} = \lambda_N^{(2)} = 0$  for all large values of  $N$ . Now fix a large  $N$  such that  $\lambda_{N+r+1}^{(1)} = \lambda_{N+1}^{(2)} = 0$ . Then we set

$$\begin{aligned} \alpha_i &= a(\lambda_{N+r-i+1}^{(1)} + i - 1) + b' && \text{for } 1 \leq i \leq N + r, \\ \beta_j &= a(\lambda_{N-j+1}^{(2)} + j - 1) && \text{for } 1 \leq j \leq N. \end{aligned}$$

We have  $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_{N+r}$ ,  $0 \leq \beta_1 < \beta_2 < \dots < \beta_N$ . Since  $N$  is large, we have  $\alpha_i = a(i - 1) + b'$  for  $1 \leq i \leq r$  and  $\beta_j = a(j - 1)$  for  $1 \leq j \leq r$ . Now we can state:

**Proposition 2.1** (Lusztig [36, 22.14]). *Recall that  $a > 0$  and  $b = ar + b'$  as above. Then  $\mathbf{a}_{\lambda} = A_N - B_N$  where*

$$\begin{aligned} A_N &= \sum_{\substack{1 \leq i \leq N+r \\ 1 \leq j \leq N}} \min(\alpha_i, \beta_j) + \sum_{1 \leq i < j \leq N+r} \min(\alpha_i, \alpha_j) + \sum_{1 \leq i < j \leq N} \min(\beta_i, \beta_j), \\ B_N &= \sum_{\substack{1 \leq i \leq N+r \\ 1 \leq j \leq N}} \min(a(i - 1) + b', a(j - 1)) \\ &\quad + \sum_{1 \leq i < j \leq N+r} \min(a(i - 1) + b', a(j - 1) + b') \\ &\quad + \sum_{1 \leq i < j \leq N} \min(a(i - 1), a(j - 1)). \end{aligned}$$

(Note that  $B_N$  only depends on  $a, b, n, N$  but not on  $\lambda$ .)

*Remark 2.2.* (a) Assume that  $a = 0$  and  $b > 0$ . Then one easily checks, directly using Hoesfmit's formulas for  $\mathbf{c}_{\lambda}$ , that

$$\begin{aligned} f_{\lambda} &\in \{\text{divisors of } n!\} && \text{for all } \lambda \in \Pi_n^2, \\ \mathbf{a}_{\lambda} &= b|\lambda^{(2)}| && \text{for all } \lambda \in \Pi_n^2. \end{aligned}$$

(If  $a = b = 0$ , then  $f_{\lambda} = |W_n|/\dim E^{\lambda}$  and  $\mathbf{a}_{\lambda} = 0$  for all  $\lambda \in \Lambda_n^2$ .)

(b) For any  $a, b \in \mathbb{N}$ , we have

$$\mathbf{a}((n), \emptyset) = 0 \quad \text{and} \quad \mathbf{a}(\emptyset, (1^n)) = L(w_0),$$

where  $w_0 \in W_n$  is the unique element of maximal length.

**Definition 2.3.** We say that  $k$  is  $L$ -good if  $f_{\lambda} 1_k \neq 0$  for all  $\lambda \in \Pi_n^2$ . In particular, any field of characteristic 0 is  $L$ -good. For fields of characteristic  $p > 0$ , the above formulas for  $f_{\lambda}$  show that the conditions are as follows.

- (i) If  $a > 0$  and  $b/a \notin \{0, 1, \dots, n-1\}$  then any field is  $L$ -good.
- (ii) If  $a > 0$  and  $b/a \in \{0, 1, \dots, n-1\}$ , then fields of characteristic  $p \neq 2$  are  $L$ -good.
- (iii) If  $a = 0$ , then fields of characteristic  $p > n$  are  $L$ -good.

(Note that the case  $a = 0$  should be merely considered as a curiosity, which may only show up in extremal situations as far as applications are concerned.)

Now let us consider the *decomposition matrix* of  $H_n$ ,

$$D = ([S^{\lambda} : D^{\mu}])_{\lambda \in \Pi_n^2, \mu \in \Lambda_n^2},$$

where  $[S^{\lambda} : D^{\mu}]$  denotes the multiplicity of  $D^{\mu}$  as a composition factor of  $S^{\lambda}$ . By Dipper–James–Murphy [10, §6], we have

$$(\Delta) \quad \begin{cases} [S^{\mu} : D^{\mu}] = 1 & \text{for any } \mu \in \Lambda_n^2, \\ [S^{\lambda} : D^{\mu}] \neq 0 & \Rightarrow \lambda \leq \mu, \end{cases}$$

where  $\leq$  denotes the dominance order on bipartitions. Note that these conditions uniquely determine the set  $\Lambda_n^2$  once the matrix  $D$  is known.

**Definition 2.4.** Let  $\beta: \Lambda_n^2 \rightarrow \Pi_n^2$  be an injective map and set  $\mathcal{B} := \beta(\Lambda_n^2) \subseteq \Pi_n^2$ . Let us denote

$$M^{\nu} := D^{\beta^{-1}(\nu)} \quad \text{for any } \nu \in \mathcal{B}.$$

We say that  $\mathcal{B}$  is “canonical basic set” for  $H_n$  if the following conditions are satisfied:

$$(\Delta_{\mathbf{a}}) \quad \begin{cases} [S^{\nu} : M^{\nu}] = 1 & \text{for any } \nu \in \mathcal{B}, \\ [S^{\lambda} : M^{\nu}] \neq 0 & \Rightarrow \lambda = \nu \text{ or } \mathbf{a}_{\nu} < \mathbf{a}_{\lambda}, \end{cases}$$

Note that the conditions  $(\Delta_{\mathbf{a}})$  uniquely determine the set  $\mathcal{B}$  and the bijection  $\beta: \Lambda_n^2 \xrightarrow{\sim} \mathcal{B}$ . Thus, we obtain a new “canonical” labelling

$$\text{Irr}(H_n) = \{M^{\nu} \mid \nu \in \mathcal{B}\}.$$

Furthermore, the submatrix

$$D^{\circ} = ([S^{\nu} : M^{\nu'}])_{\nu, \nu' \in \mathcal{B}}$$

is square and lower triangular with 1 on the diagonal, when we order the modules according to increasing values of  $\mathbf{a}_{\nu}$ . More precisely, we have a block lower triangular shape

$$D^{\circ} = \begin{pmatrix} D_0^{\circ} & & & 0 \\ & D_1^{\circ} & & \\ & & \ddots & \\ * & & & D_N^{\circ} \end{pmatrix},$$

where the block  $D_i^\circ$  has rows and columns labelled by those  $S^\nu$  and  $M^{\nu'}$ , respectively, where  $\mathbf{a}_\nu = \mathbf{a}_{\nu'} = i$ , and each  $D_i^\circ$  is the identity matrix.

**Theorem 2.5** (Geck [12], [13], [16, §6] and Geck–Rouquier [21]). *Assume that Lusztig’s conjectures (P1)–(P14) in [36, 14.2] and a certain weak version of (P15) (as specified in [16, 5.2]) hold for  $\mathbf{H}_n = \mathbf{H}_A(W_n, L)$ . Assume further that  $k$  is  $L$ -good; see Definition 2.3. Then  $H_n$  admits a canonical basic set.*

*Remark 2.6.* (a) The above result is proved by a general argument which works for Iwahori–Hecke algebras of any finite type, once the properties (P1)–(P14) and the weak version of (P15) are known to hold. This is the case, for example, when the weight function  $L$  is a multiple of the length function (the “equal parameter case”); see Lusztig [36, Chap. 15]. However, it seems to be very hard to obtain an explicit description of  $\mathcal{B}$  from the construction in the proof.

(b) If  $k$  is not  $L$ -good, it is easy to produce examples in which a canonical basic set does not exist; see [16, 4.15].

TABLE 1. Decomposition numbers for  $B_3$  with  $Q = 1$ ,  $q = -1$ .

	$[S^\lambda : D^\mu]$	$\mathbf{a}_\lambda$	$b = 0$	$b = 4$
$(3   \emptyset)$	1 . . .	$(3   \emptyset)$	0	0
$(21   \emptyset)$	. 1 . .	$(21   \emptyset)$	2	1
$(111   \emptyset)$	1 . . .	$(111   \emptyset)$	6	3
$(2   1)$	. 1 1 .	$(2   1)$	1	4
$(11   1)$	. 1 1 .	$(11   1)$	3	5
$(1   2)$	1 . . 1	$(1   2)$	1	7
$(\emptyset   3)$	. . 1 .	$(\emptyset   3)$	0	9
$(1   11)$	1 . . 1	$(1   11)$	3	10
$(\emptyset   21)$	. . . 1	$(\emptyset   21)$	2	13
$(\emptyset   111)$	. . 1 .	$(\emptyset   111)$	6	18

**Example 2.7.** Assume that  $n = 3$ ,  $Q = 1$  and  $q = -1$ . Thus, in  $H_3$ , we have the quadratic relations

$$T_t^2 = T_1 \quad \text{and} \quad T_{s_i}^2 = -T_1 - 2T_{s_i} \quad \text{for } i = 1, 2.$$

In this case, we have

$$\Lambda_3^2 = \{(3 | \emptyset), (21 | \emptyset), (2 | 1), (1 | 2)\}.$$

The decomposition matrix is printed in Table 1.

(a) Now let us take  $\xi = -1$  and write  $Q = \xi^b$ ,  $q = \xi^a$  where  $a = 1$  and  $b = 0$ . Then the canonical basic set is given by

$$\mathcal{B} = \{(3 | \emptyset), (\emptyset | 3), (1 | 2), (2 | 1)\}.$$

The bijection  $\beta: \Lambda_3^2 \rightarrow \mathcal{B}$  is given by

$$\begin{aligned} (3 \mid \emptyset) &\mapsto (3 \mid \emptyset), \\ (2 \mid \emptyset) &\mapsto (2 \mid 1), \\ (2 \mid 1) &\mapsto (\emptyset \mid 3), \\ (1 \mid 2) &\mapsto (1 \mid 2). \end{aligned}$$

- (b) Now let us take again  $\xi = -1$  and write  $Q = \xi^b$ ,  $q = \xi^a$  where  $a = 1$  and  $b = 4$ . Then we have  $\mathcal{B} = \Lambda_3^2$  and  $\beta$  is the identity.

We leave it to the reader to extract this information from the decomposition matrix printed in Table 1.

The following result was already announced in [16, 6.9], with a brief sketch of the proof. We include a more rigorous argument here.

**Theorem 2.8.** *Assume that  $b > (n-1)a > 0$ . Then  $\mathcal{B} = \Lambda_n^2$  is a canonical basic set, where the map  $\beta: \Lambda_n^2 \rightarrow \mathcal{B}$  is the identity.*

*Proof.* First of all, the formula for  $\mathbf{a}_\lambda$  in Proposition 2.1 can be simplified under the given assumptions on  $a$  and  $b$ . Indeed, let  $\lambda \in \Pi_n^2$ . Then, by [18, Example 3.6], we have

$$(*) \quad f_\lambda = 1 \quad \text{and} \quad \mathbf{a}_\lambda = b|\lambda^{(2)}| + a(n(\lambda^{(1)}) + 2n(\lambda^{(2)}) - n(\lambda^{(2)*})),$$

where we set  $n(\nu) = \sum_{i=1}^t (i-1)\nu_i$  for any partition  $\nu = (\nu_1 \geq \nu_2 \geq \dots \geq \nu_t \geq 0)$  and where  $\nu^*$  denotes the conjugate partition. Thus, any field  $k$  is  $L$ -good. The hypotheses in Theorem 2.5 concerning Lusztig's conjectures are satisfied by [18, Theorem 1.3] and [15, Corollary 7.12]. Thus, we already know that a canonical basic set  $\mathcal{B}$  exists.

In order to prove that  $\mathcal{B} = \Lambda_n^2$ , we must show that we have the following implication, for any  $\lambda \in \Pi_n^2$  and  $\mu \in \Lambda_n^2$ :

$$[S^\lambda : D^\mu] \neq 0 \quad \Rightarrow \quad \lambda = \mu \text{ or } \mathbf{a}_\mu < \mathbf{a}_\lambda.$$

Using the relation  $(\Delta)$ , we see that it is enough to prove the following implication, for any  $\lambda, \mu \in \Pi_n^2$ :

$$\lambda \leq \mu \quad \Rightarrow \quad \lambda = \mu \text{ or } \mathbf{a}_\mu < \mathbf{a}_\lambda,$$

where  $\leq$  denotes the dominance order on bipartitions. There are two cases.

*Case 1.* We have  $|\lambda^{(1)}| = |\mu^{(1)}|$  and  $|\lambda^{(2)}| = |\mu^{(2)}|$ . Then the condition  $\lambda \leq \mu$  implies that

$$\lambda^{(1)} \leq \mu^{(1)} \quad \text{and} \quad \lambda^{(2)} \leq \mu^{(2)},$$

where, on the right, the symbol  $\leq$  denotes the dominance order on partitions. Now one can argue as in the proof of “Case 1” in [18, Corollary 5.5] and conclude that  $\mathbf{a}_\mu \leq \mathbf{a}_\lambda$ , with equality only if  $\mu = \lambda$ .

*Case 2.* We have  $|\lambda^{(1)}| < |\mu^{(1)}|$  and  $|\lambda^{(2)}| > |\mu^{(2)}|$ . If  $b$  is very large with respect to  $a$ , then the formula  $(*)$  immediately shows that  $\mathbf{a}_\mu < \mathbf{a}_\lambda$ . To get this conclusion under the weaker condition that  $b > (n-1)a$ , one reduces to the case where

- $\lambda^{(1)}$  is obtained from  $\mu^{(1)}$  by decreasing one part by 1, and
- $\lambda^{(2)}$  is obtained from  $\mu^{(2)}$  by increasing one part by 1.

(This is done by an argument similar to that in [37, (I.1.16)].) Now one can argue as in the proof of “Case 2” in [18, Corollary 5.5] and compare directly the values of  $\mathbf{a}_\mu$  and  $\mathbf{a}_\lambda$ . This yields  $\mathbf{a}_\mu < \mathbf{a}_\lambda$ , as required.  $\square$

### 3. THE CASES (A) AND (B) IN THEOREM 1.1

In this section, we deal with cases (A) and (B) in Theorem 1.1, and show the existence of canonical basis sets. Throughout, we fix a weight function  $L: W_n \rightarrow \mathbb{N}$  such that

$$L(t) = b \geq 0 \quad \text{and} \quad L(s_1) = \cdots = L(s_{n-1}) = a \geq 0.$$

We also assume that  $k$  is  $L$ -good; see Definition 2.3. As before, the parameters of  $H_n$  are given by  $Q = \xi^b$  and  $q = \xi^a$ , where  $\xi \in k^\times$ . We will obtain the existence of canonical basic sets even for those values of  $a$  and  $b$  where the hypotheses of Theorem 2.5 concerning Lusztig’s conjectures are not yet known to hold.

**Theorem 3.1.** *Assume that  $k$  is  $L$ -good and that we are in case (A) of Theorem 1.1, that is, we have  $f_n(Q, q) \neq 0$ . Then*

$$\mathcal{B} = \Lambda_n^2 = \{\boldsymbol{\lambda} \in \Pi_n^2 \mid \lambda^{(1)} \text{ and } \lambda^{(2)} \text{ are } e\text{-regular}\}$$

*is a canonical basic set, where the map  $\beta: \Lambda_n^2 \rightarrow \mathcal{B}$  is the identity.*

*Proof.* As already pointed out in Theorem 1.1, we have

$$\Lambda_n^2 = \{\boldsymbol{\lambda} \in \Pi_n^2 \mid \lambda^{(1)} \text{ and } \lambda^{(2)} \text{ are } e\text{-regular}\}$$

under the given assumption on  $f_n(Q, q)$ .

Let us first deal with the case where  $a = 0$ . Since  $k$  is assumed to be  $L$ -good, we have either  $\text{char}(k) = 0$  or  $\text{char}(k) = p > n$ . In both cases,  $e > n$ . But then  $H_n$  is semisimple and  $D$  is the identity matrix; see Dipper–James [8, Theorem 5.5]. In particular,  $\mathcal{B} = \Lambda_n^2 = \Pi_n^2$  is a canonical basic set.

Let us now assume that  $a > 0$ . We shall follow the argument in the proof of [13, Prop. 6.8]. By Dipper–James [8], we can express the decomposition numbers  $[S^\lambda : D^\mu]$  of  $H_n$  in terms of decomposition numbers for Iwahori–Hecke algebras associated with the symmetric groups  $\mathfrak{S}_r$  for  $0 \leq r \leq n$ .

We need to set-up some notation. Let  $H_k(\mathfrak{S}_r, q)$  be the Iwahori–Hecke algebra of  $\mathfrak{S}_r$ , over the field  $k$  and with parameter  $q$ . By the classical results of Dipper and James [7], we have a Specht module of  $H_k(\mathfrak{S}_r, q)$  for every partition  $\lambda$  of  $r$ ; let us denote this Specht module by  $S^\lambda$ . (Again, our notation is such that  $S^\lambda$  is the Specht module labelled by  $\lambda^*$  in [7].) Furthermore, the simple modules of  $H_k(\mathfrak{S}_r, q)$  are labelled by the  $e$ -regular partitions of  $r$ ; let us denote by  $D^\mu$  the simple module labelled by the  $e$ -regular partition  $\mu$  of  $r$ . Correspondingly, we have a matrix of composition

multiplicities  $[S^\lambda : D^\mu]$ , such that the following holds:

$$(*) \quad \begin{cases} [S^\mu : D^\mu] = 1 & \text{for any } e\text{-regular partition } \mu \text{ of } r, \\ [S^\lambda : D^\mu] \neq 0 & \Rightarrow \lambda \trianglelefteq \mu, \end{cases}$$

where  $\trianglelefteq$  denotes the dominance order on partitions; see [7, Theorem 7.6].

With this notation, we have the following result for  $H_n$ ; see Dipper–James [8, Theorem 5.8]. Let  $\lambda = (\lambda^{(1)} | \lambda^{(2)}) \in \Pi_n^2$  and  $\mu = (\mu^{(1)} | \mu^{(2)}) \in \Lambda_n^2$ . If  $|\lambda^{(1)}| = |\mu^{(1)}|$  and  $|\lambda^{(2)}| = |\mu^{(2)}|$ , then

$$[S^\lambda : D^\mu] = [S^{\lambda^{(1)}} : D^{\mu^{(1)}}] \cdot [S^{\lambda^{(2)}} : D^{\mu^{(2)}}];$$

otherwise, we have  $[S^\lambda : D^\mu] = 0$ .

Now, in order to prove that  $\Lambda_n^2$  is a canonical basic set, we must show that we have the following implication:

$$[S^\lambda : D^\mu] \neq 0 \quad \Rightarrow \quad \lambda = \mu \text{ or } \mathbf{a}_\mu < \mathbf{a}_\lambda.$$

Taking into account the above results of Dipper and James, it will be enough to consider bipartitions  $\lambda = (\lambda^{(1)} | \lambda^{(2)})$  and  $\mu = (\mu^{(1)} | \mu^{(2)})$  such that  $|\lambda^{(1)}| = |\mu^{(1)}|$  and  $|\lambda^{(2)}| = |\mu^{(2)}|$ . For such bipartitions, we must show the following implication:

$$[S^{\lambda^{(1)}} : D^{\mu^{(1)}}] \neq 0 \text{ and } [S^{\lambda^{(2)}} : D^{\mu^{(2)}}] \neq 0 \quad \Rightarrow \quad \lambda = \mu \text{ or } \mathbf{a}_\mu < \mathbf{a}_\lambda.$$

Using (\*), it will actually be sufficient to prove the following implication:

$$(\dagger) \quad \lambda^{(1)} \trianglelefteq \mu^{(1)} \text{ and } \lambda^{(2)} \trianglelefteq \mu^{(2)} \quad \Rightarrow \quad \lambda = \mu \text{ or } \mathbf{a}_\mu < \mathbf{a}_\lambda.$$

Thus, we are reduced to a purely combinatorial statement. We claim that, in fact,  $(\dagger)$  holds for all bipartitions  $\lambda, \mu \in \Pi_n^2$ . To prove this, one can further reduce to the situation where  $\lambda^{(1)} = \mu^{(1)}$  or  $\lambda^{(2)} = \mu^{(2)}$ . For example, let us assume that  $\lambda^{(1)} = \mu^{(1)}$  and  $\lambda^{(2)} \trianglelefteq \mu^{(2)}$ . Then one can even further reduce to the case where  $\lambda^{(2)}$  is obtained from  $\mu^{(2)}$  by increasing one part by 1 and by decreasing another part by 1; see Macdonald [37, (I.1.16)]. But in this situation, it is straightforward to check the desired assertion by directly using the formula in Proposition 2.1. Thus,  $(\dagger)$  is proved, and this completes the proof that  $\mathcal{B} = \Lambda_n^2$  is a canonical basic set.  $\square$

Now let us turn to case (B) in Theorem 1.1. Thus, we assume that  $f_n(Q, q) = 0$  and  $q = 1$ ; note that this also gives  $Q = -1$ . Furthermore, we have

$$e = \begin{cases} \infty & \text{if } \text{char}(k) = 0, \\ p & \text{if } \text{char}(k) = p > 0. \end{cases}$$

Note also that, since  $k$  is assumed to be  $L$ -good, we must have  $b > 0$  and  $a$  cannot divide  $b$ .

We need some results concerning the complex irreducible characters of the group  $W_n$ . These characters are naturally labelled by  $\Pi_n^2$ ; we write  $\chi^\lambda$  for the irreducible character labelled by  $\lambda \in \Pi_n^2$ . (In fact, as already noted earlier, the simple modules of  $\mathbf{H}_{K,n}$  are given by the various Specht modules

$S^\lambda$ ; then  $\chi^\lambda$  is the character of the corresponding Specht module of  $W_n$  obtained by specializing  $u \mapsto 1$ .) Now consider the parabolic subgroup

$$\mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \rangle \subseteq W_n.$$

Its irreducible characters are labelled by the set  $\Pi_n$  of all partitions of  $n$ ; let us denote by  $\psi^\nu$  the irreducible character labelled by  $\nu \in \Pi_n$ . We set

$$\mathbf{a}_\nu = \sum_{i=1}^t a(i-1)\nu_i \quad \text{if } \nu = (\nu_1 \geq \nu_2 \geq \dots \geq \nu_t \geq 0).$$

(This is the  $\mathbf{a}$ -function for the Iwahori–Hecke algebra of the symmetric group with weight function given by the constant value  $a$  on each generator; see [36, 22.4].) Now, by general properties of Lusztig’s  $\mathbf{a}$ -function, there is a well-defined map (called “truncated induction” or “ $J$ -induction”)

$$J: \Pi_n \rightarrow \Pi_n^2$$

such that  $\mathbf{a}_{J(\nu)} = \mathbf{a}_\nu$  and

$$\text{Ind}_{\mathfrak{S}_n}^{W_n}(\psi^\nu) = \chi^{J(\nu)} + \text{sum of characters } \chi^\lambda \text{ such that } \mathbf{a}_\lambda > \mathbf{a}_{J(\nu)}.$$

(This could be deduced from the results in [36, Chap. 20], but Lusztig assumes the validity of (P1)–(P15) in that chapter. A reference which does not refer to these properties is provided by [14, §3].)

*Remark 3.2.* Assume that  $a = 0$ . Then  $\mathbf{a}_\nu = 0$  for all  $\nu \in \Pi_n$ . A formula for  $\mathbf{a}_\lambda$  (where  $\lambda \in \Pi_n^2$ ) is given in Remark 2.2. One easily checks that

$$J(\nu) = (\nu, \emptyset) \quad \text{for all } \nu \in \Pi_n.$$

Now assume that  $a > 0$  and write  $b = ar + b'$  as before; we have already noted above that  $b' > 0$  (since  $k$  is assumed to be  $L$ -good). Then one can give a combinatorial description of the map  $J: \Pi_n \rightarrow \Pi_n^2$ , using the results in [36, Chapter 22]. Indeed, first note that  $\chi^\nu$  is obtained by  $J$ -inducing the sign character of the Young subgroup  $\mathfrak{S}_{\nu^*}$  to  $\mathfrak{S}_n$ , where  $\nu^*$  denotes the dual partition. Thus, we have

$$\text{Ind}_{\mathfrak{S}_{\nu^*}}^{W_n}(\text{sign}) = \chi^{J(\nu)} + \text{sum of characters } \chi^\lambda \text{ such that } \mathbf{a}_\lambda > \mathbf{a}_{J(\nu)}.$$

Now write  $\nu^* = (\nu_1^* > \nu_2^* > \dots > \nu_t^* > 0)$ . Then the induction can be done in a sequence of steps, using the following embeddings:

$$\begin{aligned} \mathfrak{S}_{\nu_1^*} &\subseteq W_{\nu_1^*}, \\ W_{\nu_1^*} \times \mathfrak{S}_{\nu_2^*} &\subseteq W_{\nu_1^* + \nu_2^*}, \\ W_{\nu_1^* + \nu_2^*} \times \mathfrak{S}_{\nu_3^*} &\subseteq W_{\nu_1^* + \nu_2^* + \nu_3^*}, \\ &\vdots \\ W_{\nu_1^* + \nu_2^* + \dots + \nu_{t-1}^*} \times \mathfrak{S}_{\nu_t^*} &\subseteq W_n. \end{aligned}$$

The combinatorial rule for describing the  $J$ -induction in the above steps is provided by [36, Lemma 22.17]. For this purpose, choose a large integer  $N$  and consider the symbol

$$X_0 = \begin{pmatrix} a + b', & 2a + b', & 3a + b', & \dots, & (N + r)a + b' \\ a, & 2a, & 3a, & \dots, & Na \end{pmatrix}.$$

(This labels the trivial character of the group  $W_0 = \{1\}$ .) We define a sequence of symbols  $X_1, X_2, \dots, X_t$  as follows. Let  $X_1$  be the symbol obtained from  $X_0$  by increasing each of the  $\nu_1^*$  largest entries in  $X_0$  by  $a$ . Then let  $X_2$  be the symbol obtained from  $X_1$  by increasing each of the  $\nu_2^*$  largest entries in  $X_1$  by  $a$ , and so on. Then the symbol  $X_t$  corresponds, by the reverse of the procedure described in Section 2, to the bipartition  $J(\nu)$ .

**Example 3.3.** (i) Assume that  $a > 0$  and  $r \geq n - 1$ . Then, in each step of the construction of  $X_t$ , we will only increase entries in the top row of a symbol. Thus, we obtain

$$J(\nu) = (\nu, \emptyset) \quad \text{for all } \nu \in \Pi_n.$$

(ii) In general, let  $\nu \in \Pi_n$  and suppose that  $J(\nu) = (\lambda | \mu)$  where  $\lambda$  and  $\mu$  are partitions. Then one can check that  $\nu$  is the union of all the parts of  $\lambda$  and  $\mu$  (reordered if necessary). For example, assume that  $a = 2$  and  $b = 1$ ; then  $r = 0$  and  $b' = 1$ . We obtain

$$\begin{aligned} J(4) &= (4 | \emptyset), \\ J(31) &= (3 | 1), \\ J(22) &= (2 | 2), \\ J(211) &= (21 | 1), \\ J(1111) &= (11 | 11). \end{aligned}$$

**Theorem 3.4.** Assume that  $k$  is  $L$ -good and that we are in case (B) of Theorem 1.1, where

$$\Lambda_n^2 = \{\lambda \in \Pi_n^2 \mid \lambda^{(1)} \text{ is } e\text{-regular and } \lambda^{(2)} = \emptyset\}.$$

Then  $\mathcal{B} = \beta(\Lambda_n^2)$  is a canonical basic set for  $H_n$ , where  $\beta: \Lambda_n^2 \rightarrow \Pi_n^2$  is defined by  $(\lambda^{(1)} | \emptyset) \mapsto J(\lambda^{(1)})$ .

*Proof.* The special feature of case (B) is that, by Dipper–James [8, 5.4], the simple  $H_n$ -modules are obtained by extending (in a unique way) the simple modules of the parabolic subalgebra  $k\mathfrak{S}_n = \langle T_{s_1}, \dots, T_{s_{n-1}} \rangle_k$  to  $H_n$ . Thus, given an  $e$ -regular partition  $\nu \in \Pi_n$ , the restriction

$$D^\nu := \text{Res}_{k\mathfrak{S}_n}^{H_n}(D^{(\nu | \emptyset)})$$

is a simple module for  $k\mathfrak{S}_n$ , and all simple  $k\mathfrak{S}_n$ -modules are obtained in this way. Moreover, this yields precisely the classical labelling of the simple  $k\mathfrak{S}_n$ -modules by  $e$ -regular partitions of  $n$ .

Alternatively, we can express this by saying that the projective indecomposable  $H_n$ -modules are obtained by inducing the projective indecomposable

$k\mathfrak{S}_n$ -modules to  $H_n$ . More precisely, for any  $e$ -regular  $\nu \in \Pi_n$ , let  $P^\nu$  be a projective cover of  $D^\nu$ ; then

$$Q^\nu := \text{Ind}_{k\mathfrak{S}_n}^{H_n}(P^\nu)$$

is a projective cover of the simple  $H_n$ -module  $D^{(\nu|\emptyset)}$ . (This follows from standard results on projective modules and Frobenius reciprocity.)

It will now be convenient to use an appropriate form of Brauer reciprocity in the usual setting of modular representation theory (as in [8, Remark 5.9]). Then the multiplicity of a simple module in a Specht module is seen to be the same as the multiplicity (in the appropriate Grothendieck group) of that Specht module in the projective cover of the simple module. Thus, for any  $e$ -regular  $\nu \in \Pi_n$ , we have

$$[P^\nu] = \sum_{\lambda \in \Pi_n} [S^\lambda : D^\nu] \cdot [S^\lambda],$$

where  $[P^\nu]$ ,  $[S^\lambda]$  denote the classes of these modules in the appropriate Grothendieck group of  $k\mathfrak{S}_n$ -modules, and we have

$$[Q^\nu] = \sum_{\lambda \in \Pi_n^2} [S^\lambda : D^{(\nu|\emptyset)}] \cdot [S^\lambda],$$

where  $[Q^\nu]$ ,  $[S^\lambda]$  denote the classes of these modules in the appropriate Grothendieck group of  $H_n$ -modules.

With these preparations, let us now consider the  $\mathbf{a}$ -invariants. Using the relations  $(*)$  in the proof of Theorem 3.1, we have:

$$[P^\nu] = [S^\nu] + \text{lower terms with respect to } \leq,$$

where “lower terms” stands for a sum of classes of modules  $S^{\nu'}$  such that  $\nu' \leq \nu$  and  $\nu' \neq \nu$ . By known properties of the  $\mathbf{a}$ -function, this can also be expressed as:

$$[P^\nu] = [S^\nu] + \text{higher terms},$$

where “higher terms” stands for a sum of classes of modules  $S^{\nu'}$  such that  $\mathbf{a}_{\nu'} > \mathbf{a}_\nu$ . Now let us induce to  $H_n$ . By the definition of  $J(\nu)$ , we have

$$[\text{Ind}_{k\mathfrak{S}_n}^{H_n}(S^\nu)] = [S^{J(\nu)}] + \text{higher terms},$$

where  $\mathbf{a}_\nu = \mathbf{a}_{J(\nu)}$  and where “higher terms” stands for a sum of classes of modules  $S^{\nu'}$  such that  $\mathbf{a}_{\nu'} > \mathbf{a}_\nu$ . Furthermore, by general properties of the  $\mathbf{a}$ -function, it is known that inducing a module with a given  $\mathbf{a}$ -invariant  $i$  will result in a sum of modules of  $\mathbf{a}$ -invariants  $\geq i$ . Hence we obtain

$$\begin{aligned} [Q^\nu] &= [\text{Ind}_{k\mathfrak{S}_n}^{H_n}(S^\nu)] + \text{higher terms} \\ &= [S^{J(\nu)}] + \text{higher terms}, \end{aligned}$$

where, in both cases, “higher terms” stands for a sum of classes of modules  $S^{\nu'}$  such that  $\mathbf{a}_{\nu'} > \mathbf{a}_\nu = \mathbf{a}_{J(\nu)}$ . Thus, the conditions  $(\Delta_{\mathbf{a}})$  in Definition 2.4

are satisfied with respect to the map

$$\beta: \Lambda_n^2 \rightarrow \Pi_n^2, \quad (\nu \mid \emptyset) \mapsto J(\nu).$$

Hence,  $\mathcal{B} = \{J(\nu) \mid \nu \in \Pi_n \text{ is } e\text{-regular}\}$  is a canonical basic set.  $\square$

#### 4. THE FOCK SPACE AND CANONICAL BASES

In this section, we briefly review the deep results of Ariki and Uglov concerning the connections between the representation theory of Hecke algebras and the theory of canonical bases for quantum groups. (For general introductions to the theory of canonical bases, see Kashiwara [30] and Lusztig [35].) These results will be used in the subsequent section to describe the canonical basic set in case (C) of Theorem 1.1. The main theorems of this section are available for a wider class of algebras, namely, the Arike–Koike algebras which we now define.

**Definition 4.1.** Let  $k$  be an algebraically closed field and let  $\zeta \in k^\times$ . Let  $n, r \geq 1$  and fix parameters

$$\mathbf{u} = (u_1, \dots, u_r) \quad \text{where} \quad u_i \in \mathbb{Z}.$$

Having fixed these data, we let  $H_{n,\zeta}^{\mathbf{u}}$  be the associative  $k$ -algebra (with 1), with generators  $S_0, S_1, \dots, S_{n-1}$  and defining relations as follows:

$$\begin{aligned} S_0 S_1 S_0 S_1 &= S_1 S_0 S_1 S_0 \quad \text{and} \quad S_0 S_i = S_i S_0 \quad (\text{for } i > 1), \\ S_i S_j &= S_j S_i \quad (\text{if } |i - j| > 1), \\ S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1} \quad (\text{for } 1 \leq i \leq n-2), \\ (S_0 - \zeta^{u_1})(S_0 - \zeta^{u_2}) \cdots (S_0 - \zeta^{u_r}) &= 0, \\ (S_i - \zeta)(S_i + 1) &= 0 \quad \text{for } 1 \leq i \leq n-1. \end{aligned}$$

This algebra can be seen as an Iwahori–Hecke algebra associated with the complex reflection group  $G_{r,n} := (\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathfrak{S}_n$ . See Ariki [3, Chap. 13] and Broué–Malle [5] for further details and motivations for studying this class of algebras.

*Remark 4.2.* Let  $r = 2$ . Then we can identify  $H_{n,\zeta}^{\mathbf{u}}$  with an Iwahori–Hecke algebra of type  $B_n$ . Indeed, the generator  $S_0$  satisfies the quadratic relation

$$(S_0 - \zeta^{u_1})(S_0 - \zeta^{u_2}) = 0.$$

Then the map  $T_t \mapsto -\zeta^{-u_2} S_0$ ,  $T_{s_1} \mapsto S_1$ ,  $\dots$ ,  $T_{s_{n-1}} \mapsto S_{n-1}$  defines an isomorphism

$$H_k(W_n, -\zeta^{u_1-u_2}, \zeta) \xrightarrow{\sim} H_{2,\zeta}^{\mathbf{u}}.$$

Note that, if  $\zeta \neq 1$  and if  $u_1, u_2$  are such that  $|u_1 - u_2| \leq n-1$ , then  $f_n(-\zeta^{u_1-u_2}, \zeta) = 0$  and we are in Case (C) of Theorem 1.1.

Now Dipper–James–Mathas [9, §3] have generalized the theory of Specht modules to the algebras  $H_{n,\zeta}^{\mathbf{u}}$ ; see also Graham–Lehrer [22]. Let  $\Pi_n^r$  denote the set of all  $(r)$ -multipartitions of  $n$ , that is,  $r$ -tuples of partitions  $\boldsymbol{\lambda} = (\lambda^{(1)} \mid \dots \mid \lambda^{(r)})$  such that  $|\lambda^{(1)}| + \dots + |\lambda^{(r)}| = n$ . For any  $\boldsymbol{\lambda} \in \Pi_n^r$ , there

is a *Specht module*  $S^{\lambda, \mathbf{u}}$  for  $H_{n, \zeta}^{\mathbf{u}}$ . Each  $S^{\lambda, \mathbf{u}}$  carries a symmetric bilinear form and, taking quotients by the radical, we obtain a collection of modules  $D^{\lambda, \mathbf{u}}$ . As before, we set

$$\Lambda_n^{\mathbf{u}} := \{\lambda \in \Pi_n^r \mid D^{\lambda, \mathbf{u}} \neq \{0\}\}.$$

Then, by [9, Theorem 3.30], we have

$$\text{Irr}(H_{n, \zeta}^{\mathbf{u}}) = \{D^{\lambda, \mathbf{u}} \mid \lambda \in \Lambda_n^{\mathbf{u}}\}.$$

Furthermore, the entries of the decomposition matrix

$$D = ([S^{\lambda, \mathbf{u}} : D^{\mu, \mathbf{u}}])_{\lambda \in \Pi_n^r, \mu \in \Lambda_n^{\mathbf{u}}}$$

satisfy the conditions

$$(\Delta^{\mathbf{u}}) \quad \begin{cases} [S^{\mu, \mathbf{u}} : D^{\mu, \mathbf{u}}] = 1 & \text{for any } \mu \in \Lambda_n^{\mathbf{u}}, \\ [S^{\lambda, \mathbf{u}} : D^{\mu, \mathbf{u}}] \neq 0 & \Rightarrow \lambda \leq \mu, \end{cases}$$

where  $\leq$  denotes the dominance order on  $r$ -partitions, as defined in [9, 3.11]. Note, again, that these conditions uniquely determine the set  $\Lambda_n^{\mathbf{u}}$  once the matrix  $D$  is known. By Ariki [2], the problem of computing  $D$  (at least in the case where  $\text{char}(k) = 0$ ) can be translated to that of computing the canonical bases of a certain module over the quantum group  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$ , where  $e \geq 2$  is the order of  $\zeta$  in the multiplicative group of  $k$ .

We first give a brief overview of the results of Uglov [38] which generalize previous work of Leclerc and Thibon [33]; for a good survey on this theory, see Yvonne [39]. Let us fix an integer  $e \geq 2$ . Let  $\mathbf{u} = (u_1, \dots, u_r) \in \mathbb{Z}^r$  and let  $q$  be an indeterminate. The Fock space  $\mathfrak{F}^{\mathbf{u}}$  is defined to be the  $\mathbb{C}(q)$ -vector space generated by the symbols  $|\lambda, \mathbf{u}\rangle$  with  $\lambda \in \Pi_n^r$ :

$$\mathfrak{F}^{\mathbf{u}} := \bigoplus_{n=0}^{\infty} \bigoplus_{\lambda \in \Pi_n^r} \mathbb{C}(q) |\lambda, \mathbf{u}\rangle$$

where  $\Pi_0^r = \{\emptyset = (\emptyset, \dots, \emptyset)\}$ . Let  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  be the quantum group associated to the Lie algebra  $\widehat{\mathfrak{sl}}'_e$ . Then the deep results of Uglov show how the set  $\mathfrak{F}^{\mathbf{u}}$  can be endowed with a structure of integrable  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  module (see [38, §3.5, §4.2]). Moreover, Uglov has defined an involution  $^- : \mathfrak{F}^{\mathbf{u}} \rightarrow \mathfrak{F}^{\mathbf{u}}$ . Then one can show that there is a unique basis

$$\{G(\lambda, \mathbf{u}) \mid \lambda \in \Pi_n^r, n \in \mathbb{N}\}$$

of  $\mathfrak{F}^{\mathbf{u}}$  such that the following two conditions hold:

$$\overline{G(\lambda, \mathbf{u})} = G(\lambda, \mathbf{u}),$$

$$G(\lambda, \mathbf{u}) = |\lambda, \mathbf{u}\rangle + q\mathbb{C}[q]\text{-combination of basis elements } |\mu, \mathbf{u}\rangle.$$

The set  $\{G(\lambda, \mathbf{u})\}$  is called the Kashiwara–Lusztig *canonical basis* of  $\mathfrak{F}^{\mathbf{u}}$ .

Now we consider the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -submodule  $\mathcal{M}^{\mathbf{u}} \subseteq \mathfrak{F}^{\mathbf{u}}$  generated by  $|\emptyset, \mathbf{u}\rangle$ . It is well-known that this is isomorphic to the irreducible  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -module  $V(\Lambda)$

with highest weight

$$\Lambda := \Lambda_{u_1(\bmod e)} + \Lambda_{u_2(\bmod e)} + \cdots + \Lambda_{u_r(\bmod e)}.$$

A basis of  $\mathcal{M}^{\mathbf{u}}$  can be given by using the canonical basis of  $\mathfrak{F}^{\mathbf{u}}$  and by studying the associated crystal graph. To describe this graph, we will need some further combinatorial definitions.

Let  $\lambda = (\lambda^{(1)} \mid \cdots \mid \lambda^{(r)}) \in \Pi_n^r$  and write

$$\lambda^{(c)} = (\lambda_1^{(c)} \geq \lambda_2^{(c)} \geq \cdots \geq 0) \quad \text{for } c = 1, \dots, r.$$

The diagram of  $\lambda$  is defined as the set

$$[\lambda] := \{(a, b, c) \mid 1 \leq c \leq r, 1 \leq b \leq \lambda_a^{(c)} \text{ for } a = 1, 2, \dots\}.$$

For any “node”  $\gamma = (a, b, c) \in [\lambda]$ , we set

$$\text{res}_e(\gamma) := (b - a + u_c) \bmod e$$

and call this the  $e$ -residue of  $\gamma$  with respect to the parameters  $\mathbf{u}$ . If  $\text{res}_e(\gamma) = i$ , we say that  $\gamma$  is an  $i$ -node of  $\lambda$ .

Now suppose that  $\lambda \in \Pi_n^r$  and  $\mu \in \Pi_{n+1}^r$  for some  $n \geq 0$ . We write

$$\gamma = \mu / \lambda \quad \text{if} \quad [\lambda] \subset [\mu] \quad \text{and} \quad [\mu] = [\lambda] \cup \{\gamma\};$$

Then we call  $\gamma$  an addable node for  $\lambda$  or a removable node for  $\mu$ .

**Definition 4.3** (Foda et al. [11, p. 331]). We say that the node  $\gamma = (a, b, c)$  is “above” the node  $\gamma' = (a', b', c')$  if

- either  $b - a + u_c < b' - a' + u_{c'}$ ,
- or  $b - a + u_c = b' - a' + u_{c'}$  and  $c' < c$ .

Using this order relation on nodes, we define the notion of “good” nodes, as follows. Let  $\lambda \in \Pi_n^r$  and let  $\gamma$  be an  $i$ -node of  $\lambda$ . We say that  $\gamma$  is a *normal* node if, whenever  $\gamma'$  is an  $i$ -node of  $\lambda$  below  $\gamma$ , there are strictly more removable  $i$ -nodes between  $\gamma'$  and  $\gamma$  than there are addable  $i$ -nodes between  $\gamma'$  and  $\gamma$ . If  $\gamma$  is a highest normal  $i$ -node of  $\lambda$ , then  $\gamma$  is called a *good* node. Note that these notions heavily depend on the definition of what it means for one node to be “above” another node. These definitions (for  $r = 1$ ) first appeared in the work of Kleshchev [31] on the modular branching rule for the symmetric group; see also the discussion of these results in [32, §2].

**Definition 4.4.** For any  $n \geq 0$ , we define a subset  $\Phi_{e,n}^{\mathbf{u}} \subseteq \Pi_n^r$  recursively as follows. We set  $\Phi_{e,0}^{\mathbf{u}} = \{\emptyset\}$ . For  $n \geq 1$ , the set  $\Phi_{e,n}^{\mathbf{u}}$  is constructed as follows.

- (1) We have  $\emptyset \in \Phi_{e,n}^{\mathbf{u}}$ ;
- (2) Let  $\lambda \in \Pi_n^r$ . Then  $\lambda$  belongs to  $\Phi_{e,n}^{\mathbf{u}}$  if and only if  $\lambda / \mu = \gamma$  where  $\mu \in \Phi_{e,n-1}^{\mathbf{u}}$  and  $\gamma$  is a good  $i$ -node of  $\lambda$  for some  $i \in \{0, 1, \dots, e-1\}$ .

The set

$$\Phi_e^{\mathbf{u}} := \bigcup_{n \geq 0} \Phi_{e,n}^{\mathbf{u}}$$

will be called the set of *Uglov  $r$ -multipartitions*.

*Remark 4.5.* Let  $m \in \mathbb{Z}$  and consider the parameter set  $\mathbf{u}' = (u_1 + m, u_2 + m, \dots, u_r + m)$ . Note that a node  $\gamma$  is above a node  $\gamma'$  with respect to  $\mathbf{u}$  if and only if this holds with respect to  $\mathbf{u}'$ . It follows that  $\Phi_e^{\mathbf{u}} = \Phi_e^{\mathbf{u}'}$ .

**Theorem 4.6** (Jimbo et al. [29], Foda et al. [11], Uglov [38]). *The crystal graph of  $\mathcal{M}^{\mathbf{u}}$  has vertices labelled by the set  $\Phi_e^{\mathbf{u}}$  of Uglov  $r$ -multipartitions. Given two vertices  $\lambda \neq \mu$  in that graph, we have an edge*

$$\lambda \xrightarrow{i} \mu \quad (\text{where } 0 \leq i \leq e-1)$$

*if and only if  $\mu$  is obtained from  $\lambda$  by adding a “good”  $i$ -node*

For a general introduction to crystal graphs, see Kashiwara [30].

*Remark 4.7.* (a) Assume that  $\mathbf{u} \in \mathbb{Z}^r$  is such that

$$0 \leq u_1 \leq u_2 \leq \dots \leq u_r \leq e-1.$$

Then it is shown in Foda et al. [11, 2.11] that  $\lambda \in \Pi_{r,n}$  belongs to  $\Phi_{e,n}^{\mathbf{u}}$  if and only if the following conditions are satisfied:

- For all  $1 \leq j \leq r-1$  and  $i = 1, 2, \dots$ , we have:

$$\lambda_i^{(j+1)} \geq \lambda_{i+u_{j+1}-u_j}^{(j)} \quad \text{and} \quad \lambda_i^{(1)} \geq \lambda_{i+u_1-u_r}^{(r)};$$

- for all  $k > 0$ , among the residues appearing at the right ends of the rows of  $[\lambda]$  of length  $k$ , at least one element of  $\{0, 1, \dots, e-1\}$  does not occur.

Note that this provides a non-recursive description of the elements of  $\Lambda_n^{\mathbf{u}}$ .

(b) Assume that  $\mathbf{u} \in \mathbb{Z}^r$  is such that

$$u_1 > u_2 > \dots > u_r > 0 \quad \text{where} \quad u_i - u_{i+1} > n-1 \text{ for all } i.$$

Then the set of Uglov  $r$ -multipartitions  $\Phi_{e,n}^{\mathbf{u}}$  coincides with the set  $\mathcal{K}_{e,n}^{\mathbf{u}}$  of Kleshchev  $r$ -multipartitions as defined by Ariki [2]. More directly,  $\mathcal{K}_{e,n}^{\mathbf{u}}$  can be defined recursively in a similar way as in Definition 4.4, where we use the following order relation on nodes  $\gamma = (a, b, c)$  and  $\gamma' = (a', b', c')$ :

$$\gamma \text{ is above } \gamma' \quad \stackrel{\text{def}}{\iff} \quad c' < c \quad \text{or if} \quad c = c' \text{ and } a' < a.$$

This is the order on nodes used by Ariki [3, Theorem 10.10].

Finally, the following theorem gives a link between the canonical basis elements of  $\mathcal{M}^{\mathbf{u}}$  and the decomposition matrices of Ariki-Koike algebras. The previous results show that the canonical basis of  $\mathcal{M}^{\mathbf{u}}$  is given by

$$\{G(\mu, \mathbf{u}) \mid \mu \in \Phi_e^{\mathbf{u}}\}.$$

For each  $\mu \in \Phi_{e,n}^{\mathbf{u}}$ , we can write

$$G(\mu, \mathbf{u}) = \sum_{\lambda \in \Pi_n^r} d_{\lambda, \mu}^{\mathbf{u}}(q) |\lambda, \mathbf{u}\rangle \quad \text{where } d_{\lambda, \mu}^{\mathbf{u}}(q) \in \mathbb{C}[q].$$

With this notation, we can now state:

**Theorem 4.8** (Ariki [1],[2],[3]). *Let  $e \geq 2$ ,  $\mathbf{u} \in \mathbb{Z}^r$  and let  $H_{n,\zeta}^{\mathbf{u}}$  be the Ariki-Koike algebra over  $k$  as in Definition 4.1, where  $1 \neq \zeta \in k^\times$  is a root of unity and  $k$  has characteristic 0. Let  $e \geq 2$  be the order of  $\zeta$ . By adding multiples of  $e$  to each  $u_i$ , we may assume without loss of generality that*

$$u_1 > u_2 > \cdots > u_r > 0 \quad \text{where} \quad u_i - u_{i+1} > n - 1 \text{ for all } i.$$

*Then  $\Phi_{e,n}^{\mathbf{u}} = \mathcal{K}_{e,n}^{\mathbf{u}} = \Lambda_n^{\mathbf{u}}$  and*

$$[S^{\lambda,\mathbf{u}} : D^{\mu,\mathbf{u}}] = d_{\lambda,\mu}^{\mathbf{u}}(1)$$

*for all  $\lambda \in \Pi_n^r$  and  $\mu \in \Phi_{e,n}^{\mathbf{u}}$ .*

As a consequence, we can compute the decomposition matrices of Ariki-Koike algebras using the known combinatorial algorithm for computing canonical basis for  $\mathcal{M}^{\mathbf{u}}$ ; see Lascoux-Leclerc-Thibon [32] for  $r = 1$ , and Jacon [27] for  $r \geq 2$ .

**Corollary 4.9.** *Let us keep the same general hypotheses as in Theorem 4.8, except that we drop the condition that  $u_i - u_{i+1} > n - 1$  for all  $i$ . Then there exists a bijection  $\kappa: \Lambda_n^{\mathbf{u}} \rightarrow \Phi_{e,n}^{\mathbf{u}}$  such that*

$$[S^{\lambda,\mathbf{u}} : D^{\mu,\mathbf{u}}] = d_{\lambda,\kappa(\mu)}^{\mathbf{u}}(1)$$

*for all  $\lambda \in \Pi_n^r$  and  $\mu \in \Lambda_n^{\mathbf{u}}$ .*

*Proof.* Let  $\mathbf{u}' = (u'_1, \dots, u'_r) \in \mathbb{Z}^r$  be such that  $u'_i - u'_{i+1} > n - 1$  and  $u_i \equiv u'_i \pmod{e}$  for all  $i$ . As explained in Foda et al. [11, Note 2.7], the canonical basis  $\{G(\mu, \mathbf{u})\}$  (specialised at  $q = 1$ ) coincides with the canonical basis  $\{G(\mu, \mathbf{u}')\}$  (specialised at  $q = 1$ ), at least as far as all multipartitions  $\mu$  of total size  $\leq n$  are concerned. Hence there exists a bijection  $\kappa: \Phi_{e,n}^{\mathbf{u}'} \rightarrow \Phi_{e,n}^{\mathbf{u}}$  such that

$$d_{\lambda,\kappa(\mu)}^{\mathbf{u}}(1) = d_{\lambda,\mu}^{\mathbf{u}'}(1) \quad \text{for all } \lambda \in \Pi_n^r \text{ and } \mu \in \Phi_{e,n}^{\mathbf{u}}.$$

By Theorem 4.8, we have  $\Phi_{e,n}^{\mathbf{u}'} = \Lambda_n^{\mathbf{u}'}$  and  $[S^{\lambda,\mathbf{u}'} : D^{\mu,\mathbf{u}'}] = d_{\lambda,\mu}^{\mathbf{u}'}(1)$ . Now note that  $H_{n,\zeta}^{\mathbf{u}} = H_{n,\zeta}^{\mathbf{u}'}$  and so  $\Lambda_n^{\mathbf{u}} = \Lambda_n^{\mathbf{u}'}$ . Thus, we obtain

$$[S^{\lambda,\mathbf{u}} : D^{\mu,\mathbf{u}}] = [S^{\lambda,\mathbf{u}'} : D^{\mu,\mathbf{u}'}] = d_{\lambda,\mu}^{\mathbf{u}'}(1) = d_{\lambda,\kappa(\mu)}^{\mathbf{u}}(1)$$

for all  $\lambda \in \Pi_n^r$  and  $\mu \in \Lambda_n^{\mathbf{u}}$ , as required.  $\square$

## 5. THE CASE (C) IN THEOREM 1.1

Using the results of the previous section, we are now ready to deal with case (C) in Theorem 1.1 and show the existence of canonical basis sets when  $k$  has characteristic zero. (The case of positive characteristic remains conjectural, see Remark 5.3.) Throughout, we fix a weight function  $L: W_n \rightarrow \mathbb{N}$  as in Section 3. As before, the parameters of  $H_n$  are given by  $Q = \xi^b$  and  $q = \xi^a$ , where  $\xi \in k^\times$  and  $a, b \geq 0$ . We shall now assume that

- (C1)  $Q = -q^d$  for some  $d \in \mathbb{Z}$ ,
- (C2)  $q \neq 1$  has finite order in  $k^\times$ ,

(C3)  $\text{char}(k) \neq 2$ .

*Remark 5.1.* Assume that  $\text{char}(k) \neq 2$ ,  $f_n(Q, q) = 0$  and  $q \neq 1$ . Then  $\xi$  is a non-trivial element of finite even order in the multiplicative group of  $k$ ; furthermore, (C1), (C2) and (C3) hold.

Indeed, the condition  $f_n(Q, q) = 0$  implies that  $Q = -q^d$  for some integer  $d$  such that  $|d| \leq n - 1$ . Thus, we have

$$\xi^{b-ad} = -1 \quad \text{where } -(n-1) \leq d \leq n-1.$$

Now, if  $b \neq ad$ , then this relation shows that  $\xi$  is a non-trivial element of finite even order in  $k^\times$ . If we had  $b = ad$ , we would obtain the contradiction  $1 = \xi^{b-ad} = -1$ . Thus, the above claim is proved.

Thus, assuming that  $\text{char}(k) \neq 2$ , the conditions (C1), (C2) are somewhat weaker than the condition in case (C) of Theorem 1.1, as  $d$  is not required to be of absolute value  $\leq n - 1$ . The following results will hold assuming only (C1), (C2), (C3). We have seen above that  $\xi$  has finite order; let  $l \geq 2$  be the multiplicative order of  $\xi$ . Let  $\zeta_l \in \mathbb{C}$  be a primitive  $l$ -th root of unity. We shall consider the Iwahori–Hecke algebra

$$H_n^0 := H_{\mathbb{C}}(W_n, Q_0, q_0) \quad \text{where} \quad Q_0 := \zeta_l^b, \quad q_0 := \zeta_l^a \neq 1.$$

Note that both  $H_n$  and  $H_n^0$  are obtained by specialisation from the same generic algebra  $\mathbf{H}_n$  (defined with respect to the given weight function  $L$ ). Note also that  $f_n(Q, q) = 0 \Leftrightarrow f_n(Q_0, q_0) = 0$ , and that the parameter  $e$  defined with respect to  $q_0$  is the same as the parameter  $e$  defined with respect to  $q$  (since  $q \neq 1$ ). Now Theorem 1.1 shows that the simple modules of  $H_n$  and of  $H_n^0$  are both parametrized by the same set  $\Lambda_n^2$ . In particular, we have  $|\text{Irr}(H_n)| = |\text{Irr}(H_n^0)|$ ; see also Ariki–Mathas [4, Theorem A]. We have the following result, which reduces the determination of a canonical basic set to the case where  $k$  has characteristic zero (assuming that such basic sets exist at all).

**Lemma 5.2** (See [24, §3.1B]). *If  $H_n$  admits a canonical basic set  $\mathcal{B}$  (with respect to a map  $\beta: \Lambda_n^2 \rightarrow \Pi_n^2$ ) and  $H_n^0$  admits a canonical basic set  $\mathcal{B}^0$  (with respect to a map  $\beta^0: \Lambda_n^2 \rightarrow \Pi_n^2$ ), then we have  $\mathcal{B} = \mathcal{B}^0$  and  $\beta = \beta^0$ .*

*Remark 5.3.* In Theorem 5.4 we will determine a canonical basic set for  $H_n^0$ . If the hypotheses of Theorem 2.5 concerning Lusztig’s conjectures were known to hold in general for type  $B_n$ , then Lemma 5.2 gives a canonical basic set for  $H_n$ .

**Theorem 5.4.** *Recall that  $H_n^0 = H_{\mathbb{C}}(W_n, Q_0, q_0)$  where  $Q_0 = \zeta_l^b$  and  $q_0 = \zeta_l^a \neq 1$  are such that  $Q_0 = -q_0^d$  for some  $d \in \mathbb{Z}$ . Let  $e \geq 2$  be the multiplicative order of  $q_0$  and let  $p_0 \in \mathbb{Z}$  be such that*

$$d + p_0 e < \frac{b}{a} < d + (p_0 + 1)e.$$

(Note that the above conditions imply that  $b/a \not\equiv d \pmod{e}$ .) Then the set

$$\mathcal{B}^0 = \Phi_{e,n}^{(d+p_0e,0)}$$

is a canonical basic set for  $H_n^0$  where  $\Phi_{e,n}^{(d+p_0e,0)}$  is defined in Definition 4.4. The required map  $\beta: \Lambda_n^{\mathbf{u}} \rightarrow \Pi_n$  such that  $\mathcal{B}^0 = \beta(\Lambda_n^{\mathbf{u}})$  is given by the map  $\kappa$  in Corollary 4.9.

*Proof.* We can identify  $H_n^0$  with an Ariki–Koike algebra as in Remark 4.2. Hence, by Corollary 4.9, the decomposition matrix of  $H_n^0$  is given by the specialisation at  $q = 1$  of the canonical basis for the highest weight module  $\mathcal{M}^{\mathbf{u}}$  where  $u_1 = d + p_0e$  and  $u_2 = 0$ . Now, under the isomorphism  $H_n^0 \cong H_{2,n}^{\mathbf{u}}$ , the Specht module for  $H_n^0$  labelled by a bipartition  $\lambda$  is isomorphic to the Specht module for  $H_{2,n}^{\mathbf{u}}$  labelled by  $\lambda$ .

We will now use the same strategy as in [28] to prove the theorem. We must show that for all  $\mu \in \Phi_{e,n}^{(d+p_0e,0)}$ :

$$(*) \quad G(\mu, \mathbf{u}) = |\mu, \mathbf{u}\rangle + \sum_{\substack{\lambda \in \Pi_n^r \\ \mathbf{a}_\lambda > \mathbf{a}_\mu}} d_{\lambda, \mu}(q) |\lambda, \mathbf{u}\rangle,$$

Again, to prove (\*), it is sufficient to show that the matrix of the involution on the Fock space  $\mathfrak{F}^{\mathbf{u}}$  is lower unitriangular with respect to the  $\mathbf{a}$ -value (see [28, Theorem 4.6]). Hence, we want to show that for all  $\lambda \in \Pi_n^2$ , we have:

$$(**) \quad \overline{|\lambda, \mathbf{u}\rangle} = |\lambda, \mathbf{u}\rangle + \text{sum of } |\mu, \mathbf{u}\rangle \text{ with } \mathbf{a}_\lambda < \mathbf{a}_\mu.$$

The proof of (\*\*) is rather long, but it is entirely analogous to the proof of [28, Theorem 4.6]. We only give the main arguments needed in this proof.

First, note that the formula in Section 2 shows how we can compute the values  $\mathbf{a}_\lambda$ . Put  $m^{(1)} = b/a$  and  $m^{(2)} = 0$ . Let  $\lambda \in \Pi_n^2$ ,  $\mu \in \Pi_{n+1}^2$  and  $\nu \in \Pi_{n+1}^2$  and assume that there exists nodes  $\gamma_1 = (a_1, b_1, c_1)$  and  $\gamma_2 = (a_2, b_2, c_2)$  such that

$$[\mu] = [\lambda] \cup \{\gamma_1\} \quad \text{and} \quad [\nu] = [\lambda] \cup \{\gamma_2\}.$$

Assume in addition that we have:

$$\lambda_{a_1}^{(c_1)} - a_1 + m^{(c_1)} > \lambda_{a_2}^{(c_2)} - a_2 + m^{(c_2)}.$$

Then it is easy to see that  $\mathbf{a}_\nu > \mathbf{a}_\mu$  (we have a similar property when  $a$  divides  $b$  in [28, Proposition 4.3]).

Now, Let  $\lambda \in \Pi_n^2$ . Then the decomposition of  $\overline{|\lambda, \mathbf{u}\rangle}$  as a linear combination of  $|\mu, \mathbf{u}\rangle$  with  $\mu \in \Pi_n^2$  can be obtained by using certain rules defined by Uglov [38, Proposition 3.16]. These rules show that a bipartition  $|\mu, \mathbf{u}\rangle$  appearing in the decomposition of  $\overline{|\lambda, \mathbf{u}\rangle}$  is obtained from another bipartition  $|\nu, \mathbf{u}\rangle$ , which is known by induction, by removing a ribbon  $R$  in  $\mu$  and adding a ribbon  $R'$  of same size in the resulting bipartition. We want to show that  $\mathbf{a}_\nu < \mathbf{a}_\mu$  and the result will follow by induction. Assume that the foot (that is the bottom-right most square) of  $R$  is on the part  $\nu_{j_1}^{(i_1)}$  and that

the foot of the ribbon  $R'$  is on the part  $\mu_{j_2}^{(i_2)}$ . Then the key property is the following:

- if  $i_1 = 2$  and  $i_2 = 1$ , we have  $\nu_{j_1}^{(i_1)} - j_1 \geq \nu_{j_2}^{(i_2)} - j_2 + d + (p_0 + 1)e$ ,
- if  $i_1 = 1$  and  $i_2 = 2$ , we have  $\nu_{j_1}^{(i_1)} - j_1 + d + p_0e \geq \nu_{j_2}^{(i_2)} - j_2$ ,
- if  $i_1 = i_2$ , we have  $\nu_{j_1}^{(i_1)} - j_1 > \nu_{j_2}^{(i_2)} - j_2$ .

The proof of this property is obtained by studying Uglov rules [38] and is analogous to the proof of [28, Lemma 4.5]. Now, since

$$0 < \frac{b}{a} - (d + p_0e) < e,$$

we obtain:

$$\nu_{j_1}^{(i_1)} - j_1 + m^{(i_1)} > \nu_{j_2}^{(i_2)} - j_2 + m^{(i_2)}.$$

Then, we can conclude by induction exactly as in [28, Section 4.B].  $\square$

**Example 5.5.** Assume that we have  $a = 1$ . Then  $l = e \geq 2$  must be an even number and we can take  $d = b + e/2$ . Then we have:

$$b + \frac{e}{2} - e < \frac{b}{a} < b + \frac{e}{2}$$

and so  $p_0 = -1$ . Hence, by Theorem 5.4 and Remark 4.5, the set

$$\mathcal{B}^0 = \Phi_{e,n}^{(b-e/2,0)} = \Phi_{e,n}^{(b,e/2)}$$

is a canonical basic set for  $H_n^0$ .

In particular, in the case  $b = 1$  (the “equal parameter case”), we have  $\mathcal{B}^0 = \Phi_{e,n}^{(1,e/2)}$ , the set of “FLOTW bipartitions” as in Remark 4.7(a). Thus, we recover the result shown in [26]. If  $b = 0$ , we obtain  $\mathcal{B}^0 = \Phi_{e,n}^{(0,e/2)}$ , and we recover the result shown in [25].

**Example 5.6.** Assume that  $a = 2$  and that there exists a nonnegative integer  $r$  such that  $b = 2r + 1$ . Then  $l$  must be even and  $e = l/2$  is an odd number. Then we have  $d \equiv r + (1 - e)/2 \pmod{e}$ . Now, since

$$r + (1 - e)/2 < r + 1/2 < r + (1 + e)/2,$$

Theorem 5.4 implies that  $\mathcal{B}^0 = \Phi_{e,n}^{(r+(1-e)/2,0)}$  is a canonical basic set for  $H_n^0$ .

**Example 5.7.** Assume that we have  $b > a(n - 1) + e$ . Then we have:  $d + p_0e > n - 1$ . Hence the canonical basic set  $\mathcal{B}^0 = \Phi_{e,n}^{(b-e/2,0)}$  for  $H_n^0$  coincides with the set of Kleshchev bipartitions by Remark 4.7. Note that in this case, we have  $b > a(n - 1)$ , which was already dealt with in Theorem 2.8.

*Remark 5.8.* Following [28], it would be possible to state a version of Theorem 5.4 which is valid for an Ariki–Koike algebra  $H_{n,\zeta}^{\mathbf{a}}$  as in Definition 4.1 where  $r \geq 3$ . The  $\mathbf{a}$ -invariants in this case are derived from the Schur elements as computed by Geck–Iancu–Malle [19]. We omit further details.

## REFERENCES

- [1] S. ARIKI, On the decomposition numbers of the Hecke algebra of  $G(m, 1, n)$ , J. Math. Kyoto Univ. **36** (1996), 789–808.
- [2] S. ARIKI, On the classification of simple modules for cyclotomic Hecke algebras of type  $G(m, 1, n)$  and Kleshchev multipartitions, Osaka J. Math. **38** (2001), 827–837.
- [3] S. ARIKI, Representations of quantum algebras and combinatorics of Young tableaux, University Lecture Series **26**, Amer. Math. Soc., Providence, RI, 2002.
- [4] S. ARIKI AND A. MATHAS, The number of simple modules of the Hecke algebras of type  $G(r, 1, n)$ , Math. Z. **233** (2000), 601–623.
- [5] M. BROUÉ AND G. MALLE, Zyklotomische Heckealgebren, Astérisque **212** (1993), 119–189.
- [6] R. DIPPER, M. GECK, G. HISS AND G. MALLE, Representations of Hecke algebras and finite groups of Lie type. *In*: Algorithmic algebra and number theory (Heidelberg, 1997), pp. 331–378, Springer Verlag, Berlin/Heidelberg, 1998.
- [7] R. DIPPER AND G. D. JAMES, Representations of Hecke algebras of general linear groups, Proc. London Math. Soc. **52** (1986), 20–52.
- [8] R. DIPPER AND G. D. JAMES, Representations of Hecke algebras of type  $B$ , J. Algebra **146** (1992), 454–481.
- [9] R. DIPPER, G. D. JAMES AND A. MATHAS, Cyclotomic  $q$ -Schur algebras, Math. Z. **229** (1998), 385–416.
- [10] R. DIPPER, G. D. JAMES AND G. E. MURPHY, Hecke algebras of type  $B_n$  at roots of unity, Proc. London Math. Soc. **70** (1995), 505–528.
- [11] O. FODA, B. LECLERC, M. OKADO, J.-Y. THIBON AND T. WELSH, Branching functions of  $A_{n-1}^{(1)}$  and Jantzen-Seitz problem for Ariki-Koike algebras, Advances in Math. **141** (1999), 322–365.
- [12] M. GECK, Kazhdan-Lusztig cells and decomposition numbers. Represent. Theory **2** (1998), 264–277 (electronic).
- [13] M. GECK, On the representation theory of Iwahori-Hecke algebras of extended finite Weyl groups. Represent. Theory **4** (2000), 370–397 (electronic).
- [14] M. GECK, Constructible characters, leading coefficients and left cells for finite Coxeter groups with unequal parameters, Represent. Theory **6** (2002), 1–30 (electronic).
- [15] M. GECK, Relative Kazhdan-Lusztig cells (*submitted*); preprint available at <http://arXiv.org/math.RT/0504216>
- [16] M. GECK, Modular representations of Hecke algebras, EPFL Press, *to appear*; preprint available at <http://arXiv.org/math.RT/0511548>.
- [17] M. GECK, Modular principal series representations (*submitted*); preprint available at <http://arXiv.org/math.RT/0603046>.
- [18] M. GECK AND L. IANCU, Lusztig’s  $a$ -function in type  $B_n$  in the asymptotic case, Nagoya J. Math. *to appear*; preprint at <http://arXiv.org/math.RT/0504213>.
- [19] M. GECK, L. IANCU AND G. MALLE, Weights of Markov traces and generic degrees, Indag. Mathem., N. S., **11** (2000), 379–397.
- [20] M. GECK AND G. PFEIFFER, Characters of finite Coxeter groups and Iwahori-Hecke algebras, London Math. Soc. Monographs, New Series **21**, Oxford University Press, New York 2000. xvi+446 pp.
- [21] M. GECK AND R. ROUQUIER, Filtrations on projective modules for Iwahori-Hecke algebras. *In*: Modular Representation Theory of Finite Groups (Charlottesville, VA, 1998; eds. M. J. Collins, B. J. Parshall and L. L. Scott), p. 211–221, Walter de Gruyter, Berlin 2001.
- [22] J. J. GRAHAM AND G. I. LEHRER, Cellular algebras, Invent. Math. **123** (1996), 1–34.
- [23] P. N. HOEFSMIT, Representations of Hecke algebras of finite groups with BN pairs of classical type, Ph.D. thesis, University of British Columbia, Vancouver, 1974.

- [24] N. JACON, Représentations modulaires des algèbres de Hecke et des algèbres de Ariki-Koike, Ph. D. thesis, Université Lyon 1, 2004; available at “theses-ON-line” <http://tel.ccsd.cnrs.fr/documents/archives0/00/00/63/83>.
- [25] N. JACON, Sur les représentations modulaires des algèbres de Hecke de type  $D_n$ , J. Algebra **274** (2004), 607–628.
- [26] N. JACON, On the parametrization of the simple modules for Ariki-Koike algebras at roots of unity, J. Math. Kyoto Univ. **44** (2004), 729–767.
- [27] N. JACON, An algorithm for the computation of the decomposition matrices for Ariki-Koike algebras, J. Algebra **292** (2005), 100–109.
- [28] N. JACON, Crystal graphs of higher level  $q$ -deformed Fock spaces, Lusztig  $a$ -values and Ariki-Koike algebras, Algebras and Represent. Theory (*to appear*); preprint available at [arXiv/math.RT/0504267](http://arxiv.org/abs/math.RT/0504267).
- [29] M. JIMBO, K. C. MISRA, T. MIWA AND M. OKADO, Combinatorics of representations of  $U_q(\hat{\mathfrak{sl}}(n))$  at  $q = 0$ , Comm. Math. Phys. **136** (1991), 543–566.
- [30] M. KASHIWARA, Bases cristallines des groupes quantiques, Cours Spécialisés **9**, Soc. Math. France, Paris, 2002.
- [31] A. KLESHCHEV, Branching rules for modular representations of symmetric groups I, J. Algebra **178** (1995), 493–511; II, J. reine angew. Math. **459** (1995), 163–212.
- [32] A. LASCoux, B. LECLERC AND J. Y. THIBON, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, Comm. Math. Physics **181** (1996), 205–263.
- [33] B. LECLERC AND J. Y. THIBON, Canonical basis of  $q$ -deformed Fock spaces, Int. Math. Res. Notices, **9** (1996), 447–456.
- [34] G. LUSZTIG, Characters of reductive groups over a finite field, Annals Math. Studies, vol. 107, Princeton University Press, 1984.
- [35] G. LUSZTIG, Introduction to quantum groups, Progress in Math. **110**, Birkhäuser, Boston, 1993.
- [36] G. LUSZTIG, Hecke algebras with unequal parameters, CRM Monographs Ser. **18**, Amer. Math. Soc., Providence, RI, 2003.
- [37] I. G. MACDONALD, Symmetric functions and Hall polynomials (second edition), Oxford University Press, 1995.
- [38] D. UGLOV, Canonical bases of higher-level  $q$ -deformed Fock spaces and Kazhdan–Lusztig polynomials; Physical combinatorics (Kyoto, 1999), 249–299; Progress in Math. **191**, Birkhäuser, Boston, 2000.
- [39] X. YVONNE, Bases canoniques d’espaces de Fock en niveau supérieur, Ph. D. thesis, Université de Caen, 2005.

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